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Resonant scattering by a harbor with two coupled basins

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SUMMARY
A harbor with two coupled rectangular basins is subjected to periodic incident waves. Ignoring friction the scattering problem is solved by the method of matched asymptotics for narrow junctions. The example of two identical basins is analyzed in detail for the resonant spectrum and response. It is shown that for certain modes the inner basin is less shielded.

1. Introduction

Since the paper of Miles and Munk [1] who first treated harbor oscillations by a scattering theory, the study of harbor resonance has been steadily progressing both theoretically and experimentally. In the context of infinitesimal waves over constant depth, effective numerical methods have been worked out for harbors with a straight coast but arbitrary basin form (Hwang and Tuck [2], Lee [3], and Su [4]). For variable but shallow depth, a versatile hybrid finite element method has recently been devised by Chen and Mei [5] for harbors with arbitrary basin form and quite general coastline or breakwater configuration. Thus insofar as the engineering needs of computational tools are concerned, the linearized long wave theory is in a rather satisfactory state.

The analytical aspects of harbor theory, however, have been thoroughly studied only for a harbor with one basin ([1], Garrett [6] and Miles [7]) or a basin with an entry channel ([7], Carrier, Shaw and Miyata [8]) and partially studied for the Helmholtz modes in coupled basins by Miles and Lee [9]. Lee and Raichlen [10] and [11] recently examined numerically and experimentally a harbor with two circular basins of equal areas coupled through a narrow passage. Two interesting features have been revealed. One is that the resonant peaks on the plot of amplification factor vs. wave number are doubled in density and appear in pairs, in comparison with the case of one basin. The other is that, if the line of centers of the two basins is normal to the coast, the (lower/higher) resonant mode of the pair i.e., the mode with the (smaller/larger) resonant frequency, gives rise to greater maximum response in the (inner/outer) basin. Moreover, the maximum at the lower mode is greater than that at the higher mode, see Fig. 7 of [10], Figs. 16 and 26 of [11]. This implies the interesting possibility that the inner basin can be less protected. This paper aims at deducing these and other features analytically.

To achieve these aims we restrict ourselves to constant depth and basins of rectangular form so that analytical solutions can be explicitly obtained. We shall make a further simplifying assumption that the junction widths are small compared to the basin size and the wavelength of interest. By using the method of matched asymptotic approximations similar to Buchwald [12] and Tuck [13], the solution for general basin dimensions is given in Section 2. The method can be applied to other geometries which are combinations of rectangular basins and narrow channels, see Ünlüata and Mei [14].

The main purpose of this paper is the deduction of physical implications. The special example of two equal basins of which the line of centers is normal to a straight coast is discussed in some detail. The spectrum of the resonant modes is first analyzed in Section 3. Of particular interest are the effects of junction widths on the pair of modes closely associated with the natural mode k_{nm} of a closed constituent basin. The resonant responses of the most important and the first few modes are discussed in Section 4. In Section 5 we discuss how the present results are extended to include the effect of junctions of finite thickness. Sample numerical results are given in Section 6 which confirm the analytical conclusions, and are consistent with the work of Lee and Raichlen.

2. Approximate solution for small ka

Let the motion be simple harmonic in time so that the free surface displacement can be written as

$$\eta(x, y, t) = \mathrm{Re}\{\zeta(x, y)e^{j\omega t}\}. \tag{2.1}$$

The governing equation for ζ is

$$(\nabla^2 + k^2)\zeta = 0 \tag{2.2}$$

where $\omega = (gh)^{\frac{1}{2}}k$ in shallow water.

In the far field away from the junctions, the horizontal length scale is the wavelength ($\sim 1/k$). For small junctions $ka \ll 1$, the scattered wave field can be approximated by a point source centered at the junction. Assume the incident wave to have amplitude A and to approach the coast perpendicularly. The one-term outer solution for the ocean is just the source solution superposed on the incident and reflected waves, or

$$\zeta_0 = 2A\cos kx - \frac{j\omega}{g} Q_0\left(\frac{j}{2} H_0^{(2)}(kr)\right), \quad x > 0, \tag{2.3}$$

where $r^2 = x^2 + y^2$, and $H_0^{(2)}$ is the Hankel function of the second kind. Q_0 is the unknown discharge* at the harbor mouth. For simplicity the (x, y) coordinates are centered at the junction J (Fig. 2.1).

For the basins, it is convenient to express the source solutions in a universal form. Thus, for each source we shall use a coordinate system with the origin at a corner such that the basin is in the first quadrant (Fig. 2.1). The one term outer solutions are

* In this paper, all discharges refer to a unit depth of fluid.

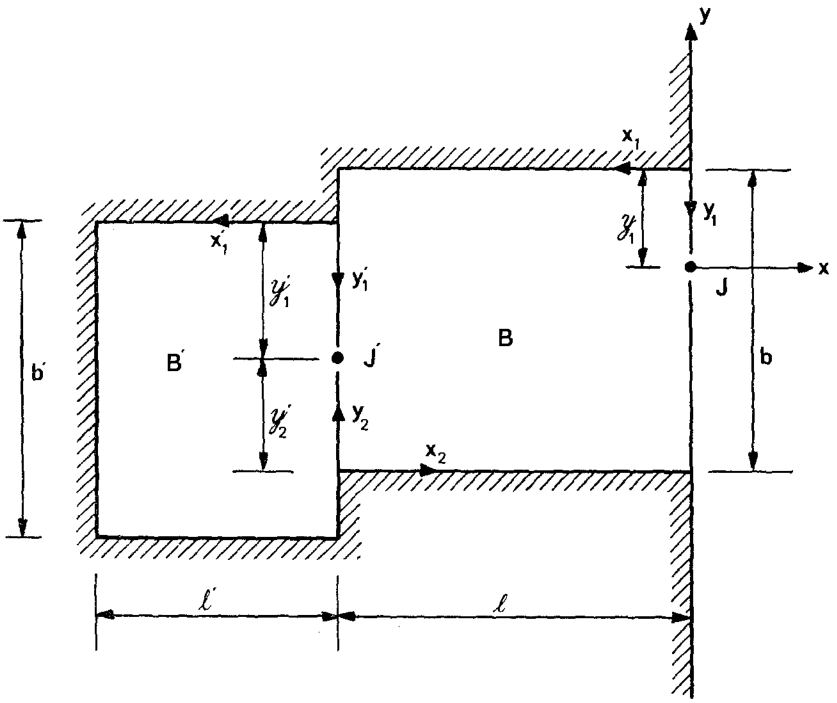

Figure 2.1. Definition sketch and coordinate systems.

$$\text{Basin } B\text{: } \zeta_B = -\frac{j\omega}{g}[Q_1G_1 + Q_2G_2], \tag{2.4a}$$

$$\text{Basin } B'\text{: } \zeta_{B'} = -\frac{j\omega}{g}Q_1'G_1'. \tag{2.4b}$$

The discharges Q_1, Q_1' and Q_2 are to be found. The Green's function for a unit source on the boundary has been given in [1] and is quoted here for convenience. The explicit solution for basin B is

$$G_\alpha \equiv G(x_\alpha, y_\alpha|y_\alpha) = \sum_{n=0}^{\infty} X_n(x_\alpha)Y_n(y_\alpha)Y_n(y_\alpha), \quad \alpha = 1, 2, \tag{2.5a}$$

with*

$$X_n(x_\alpha) \equiv \frac{\varepsilon_n \cos K_n(x_\alpha - l)}{K_n b \sin K_n l}, \tag{2.5b}$$

$$Y_n(y_\alpha) = \cos(n\pi y_\alpha/b), \tag{2.5c}$$

$$K_n = [k^2 - (n\pi/b)^2]^{\frac{1}{2}}, \tag{2.5d}$$

where ε_n is the Jacobi symbol: $\varepsilon_0 = 1$, $\varepsilon_n = 2$, $n = 1, 2, 3, \ldots$ and y_α is the center of the junction α.

* The notation Y_n should not be confused with Weber's Bessel function.

For the basin B', G_1' is defined similarly by replacing x_1', y_1', y_1', l', b' for their unprimed counterparts above. In particular, we have,

$$K_n' = [k^2 - (n\pi/b')^2]^{\frac{1}{2}}. \tag{2.6}$$

For small kr, the two term inner expansion of the outer solution ζ_0 in the neighborhood of the junction J is straightforward and is

$$\zeta_0 \cong 2A - j\frac{\omega}{g} Q_0\left(\frac{j}{2} + \frac{1}{\pi}\ln\gamma\frac{k}{2}\right) - j\frac{\omega}{g}\frac{Q_0}{\pi}\ln r + O(kr\ln kr),$$
$$x > 0, \quad \text{right of } J, \tag{2.7}$$

where $\ln\gamma = 0.5772157 =$ Euler's constant. The two-term inner expansions of the outer solutions ζ_B, ζ_B', are somewhat lengthly, and are obtained by first performing partial summation of a series in order to produce a logarithmic term which represents the singular part. We leave the details to Appendix A and only quote the results here:

$$\zeta_B \cong -j\frac{\omega}{g} Q_1\left[\frac{1}{\pi}\ln\left(\frac{2\pi}{b}\sin\frac{\pi y_1}{b}\right) + F_{1,1}\right] - j\frac{\omega}{g} Q_2 G_{2,1} - j\frac{\omega}{g}\frac{Q_1}{\pi}\ln r_1,$$
$$x_1 > 0, \quad \text{left of } J; \tag{2.8a}$$

$$\zeta_B \cong -j\frac{\omega}{g}[Q_1 G_{1,2}] - j\frac{\omega}{g} Q_2\left[\frac{1}{\pi}\ln\left(\frac{2\pi}{b}\sin\frac{\pi y_2}{b}\right) + F_{2,2}\right] - j\frac{\omega}{g}\frac{Q_2}{\pi}\ln r_2,$$
$$x_2 > 0, \quad \text{right of } J' \tag{2.8b}$$

where J' denotes the inter-basin junction, and

$$\zeta_{B'} \cong -j\frac{\omega}{g} Q_1'\left[\frac{1}{\pi}\ln\left(\frac{2\pi}{b'}\sin\frac{\pi y_1'}{b'}\right) + F_{1,1}'\right] - j\frac{\omega}{g}\frac{Q_1'}{\pi}\ln r_1',$$
$$x_1' > 0, \quad \text{left of } J' \tag{2.9a}$$

with

$$r_\alpha^2 = x_\alpha^2 + (y_\alpha - y_\alpha)^2, \quad \alpha = 1, 2; \qquad r_1'^2 = x_1'^2 + (y_1' - y_1')^2. \tag{2.9b}$$

This approximation is derived under the restriction that the junctions must be far from the corners. This restriction is assumed throughout our paper and the modification otherwise needed is discussed in [14]. $G_{1,2}$ represents the intensity of the Green's function G_1 as felt at the junction J':

$$G_{1,2} \equiv G(x_1 = l, y_1 = b - y_2|y_1) = \sum_{n=0}^{\infty}\frac{\varepsilon_n Y_n(y_1)Y_n(b - y_2)}{K_n b\sin K_n l}. \tag{2.10}$$

By definition of Y_n, $Y_n(y_1)Y_n(b - y_2) = Y_n(b - y_1)Y_n(y_2)$; it follows that

$$G_{2,1} = G(x_2 = l, y_2 = b - y_1|y_2) = G_{1,2} \tag{2.11}$$

which is simply the principle of reciprocity.

Lastly $F_{\alpha,\alpha}$ is the regular part of the series of G_α evaluated at the source point α itself:

$$F_{\alpha,\alpha} = \frac{\cot kl}{kb} + \sum_{n=1}^{\infty} 2\left(\frac{\cot K_n l}{K_n b} + \frac{1}{n\pi}\right) Y_n^2(y_\alpha). \tag{2.12}$$

$F'_{1,1}$ is defined similarly with b, l, K_n, y_1 replaced by b', l', K'_n, and y'_1 respectively.

The near field observer feels the presence of the junction and the adjoining breakwaters, but not other sides of the basins. Since the local length scale is a, the governing equation is approximately Laplacian $\nabla^2\zeta = O(ka)^2$. Thus we consider the potential flow past a typical slit in an infinitely long thin barrier. This inner problem has been studied by Tuck [13] from which the following outer expansions of the two term inner solutions may be inferred. We have for r/a_1, $r_1/a_1 \gg 1$

$$\zeta_J \cong C_1 \mp M_1 \ln\frac{a_1}{2} + M_1\begin{pmatrix} \ln r \\ -\ln r_1 \end{pmatrix}, \qquad \begin{cases} x > 0 \\ x_1 > 0 \end{cases} \tag{2.13}$$

and for r_2/a_2, $r'_1/a_2 \gg 1$,

$$\zeta_J \cong C_1 \mp M_2 \ln\frac{a_2}{2} + M_2\begin{pmatrix} \ln r_2 \\ -\ln r'_1 \end{pmatrix}, \qquad \begin{cases} x_2 > 0 \\ x'_1 > 0 \end{cases} \tag{2.14}$$

The omitted terms are of $O(a/r)^2$.

The unknown coefficients are Q_0, Q_1, Q_2, Q'_1, M_1, M_2, C_1 and C_2. By matching the constant terms and the logarithmic terms in four intermediate regions (both sides of the two junctions), eight algebraic equations are obtained. They are easily solved to give:

$$\pi M_1 = -\frac{j\omega Q_0}{g} = \frac{j\omega Q_1}{g} = -2A\left[\frac{j}{2} + F_{1,1} - I - \frac{G_{1,2}^2}{W}\right]^{-1}, \tag{2.15a}$$

$$\pi M_2 = \frac{j\omega Q'_1}{g} = \frac{-j\omega Q_2}{g} = \pi M_1\, G_{1,2}/W \tag{2.15b}$$

where

$$W \equiv F'_{1,1} + F_{2,2} - I', \tag{2.16}$$

$$I = \frac{1}{\pi}\ln\left[4b\left(\pi\gamma k a_1^2 \sin\frac{\pi y_1}{b}\right)^{-1}\right] \tag{2.17}$$

and

$$I' = \frac{1}{\pi}\ln\left\{bb'\left[\pi^2 a_2^2\left(\sin\frac{\pi y'_1}{b'}\right)\left(\sin\frac{\pi y_2}{b}\right)\right]^{-1}\right\}. \tag{2.18}$$

The constants C_1 and C_2 can be easily obtained but will be omitted here since they do not appear in the outer solutions on which all subsequent discussions will be focused. We remark that I depends on k while I' does not.

With the discharges found the outer solutions are complete. Substitution into (2.3), (2.4a) and (2.4b) gives the response of the ocean and harbor regions at any point far away from the junctions.

Instead of the local responses as represented by the preceeding equations, it is convenient to refer to the normalized mean-square response for each basin,

$$\sigma_B^2 \equiv \frac{1}{bl}\frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt \int_0^l dx_1 \int_0^b dy_1 [\mathrm{Re}(\zeta_B e^{j\omega t}/2A)]^2$$

$$= \frac{1}{4}\left|\frac{\pi M_1}{2A}\right|^2 \left[E_1 - \frac{2G_{1,2}}{W}E_{1,2} + \left(\frac{G_{1,2}}{W}\right)^2 E_2\right], \tag{2.19a}$$

$$\sigma_{B'}^2 \equiv \frac{1}{b'l'}\frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt \int_b^{l'} dx_1' \int_0^{b'} dy_1' [\mathrm{Re}(\zeta_{B'} e^{j\omega t}/2A)]^2$$

$$= \frac{1}{4}\left|\frac{\pi M_1}{2A}\right|^2 \left(\frac{G_{1,2}}{W}\right)^2 E_1' \tag{2.19b}$$

where

$$E_\alpha = \sum_{n=0}^{\infty} \frac{\varepsilon_n Y^2{}_n(y_\alpha)}{(K_n b \sin K_n l)^2}\left[1 + \frac{\sin 2K_n l}{2K_n l}\right], \qquad \alpha = 1, 2, \tag{2.20a}$$

$$E_{1,2} = \sum_{n=0}^{\infty} \frac{\varepsilon_n Y_n(y_1) Y_n(b - y_2)}{(K_n b \sin K_n l)^2}\left(\cos K_n l + \frac{\sin K_n l}{K_n l}\right). \tag{2.20b}$$

E_1' is defined similarly as E_1 with K_n, b, l, y_1 replaced by K_n', b', l', y_1' respectively. We omit the straightforward details which require the use of orthogonality of Y_n.

From these formulas numerical results can be obtained simply and will be presented in Section 6. The case in which the two mouths J and J' are on adjacent sides of the outer basin can be similarly treated but is omitted here. We now seek analytical implications.

3. Resonant wave number spectrum of the harbor with coupled basins

Although k is always real, the poles in the complex k-plane of the scattered wave amplitude, i.e., of the discharge M_1 through the harbor mouth, correspond to the resonant peaks of the forced oscillation in the basins. The closer the pole is to the real k axis, the greater the peak response. Resorting to the complex k-plane is a well-known mathematical artifice in other branches of physics, e.g., quantum scattering theory, see Sitenko [15], and has also been used in water waves, Longuet-Higgins [16]. One must however not conclude that the discharges M_1 and M_2 will actually be infinite, since k is physically never complex.

From (2.15a, b) it is evident that the roots of the complex equation

$$\frac{j}{2} + F_{1,1} - I - \frac{G_{1,2}^2}{W} = 0 \tag{3.1}$$

correspond to these poles. Let a typical root be denoted as

$$k = \tilde{k} + j\hat{k}. \tag{3.2}$$

If the imaginary part $\hat{k}$ is small, it gives the rate of radiation damping, the existence of which renders the peak amplitudes finite. The real part $\tilde{k}$ is then the resonant wave number.

For narrow J one expects that

$$\hat{k}/\tilde{k} \ll 1, \tag{3.3}$$

so that (3.1) can be expanded in the form

$$\frac{j}{2} + \left[F_{1,1} - I - \frac{G_{1,2}^2}{W}\right]_{\tilde{k}} + j\hat{k}\left[\frac{d}{dk}\left(F_{1,1} - I - \frac{G_{1,2}^2}{W}\right)\right]_{\tilde{k}} \cong 0 \tag{3.4}$$

where $[\,]_{\tilde{k}}$ indicates that the quantity in the bracket is to be evaluated at $k = \tilde{k}$. Equating the real and imaginary parts to zero separately we have

$$F_{1,1} - I - \frac{G_{1,2}^2}{W} = 0, \qquad k = \tilde{k}, \tag{3.5}$$

$$\frac{\hat{k}}{\tilde{k}} = -\frac{1}{2}\left[k\frac{d}{dk}\left(F_{1,1} - I - \frac{G_{1,2}^2}{W}\right)\right]_{\tilde{k}}^{-1}. \tag{3.6}$$

The first task is to solve (3.5) for $\tilde{k}$. For brevity we restrict our attention to the case of two equal basins: $b = b'$, $l = l'$. Furthermore, the junctions are assumed to be at the centers of the sides on which they lie, i.e., $y_1 = y_2 = y_1' = b/2$. This special case also turns out to be the most interesting.

Non-Helmholtz modes (n *or* $m \neq 0$)

An immediate consequence of symmetry is that n is even. For narrow junctions the resonant peaks must be close to the natural modes of the closed constituent basins k_{nm}. Let

$$k - k_{nm} = \Delta, \qquad \Delta \ll k_{nm}. \tag{3.7}$$

By Taylor expansion of (2.5d) and (2.6) and taking only the dominant terms of the series (2.12) and (2.10) it can be shown that

$$F_{1,1} \cong F_{1,1}' \cong F_{2,2} \cong \frac{c}{\Delta}, \quad G_{1,2} \cong \frac{c}{\Delta}\,\frac{\cos n\pi}{\cos m\pi} \quad \text{and} \quad W \cong \frac{2c}{\Delta} - I' \tag{3.8}$$

where

$$c = \frac{\varepsilon_n \varepsilon_m}{2k_{nm} bl}. \tag{3.9}$$

Equation (3.5) becomes approximately

$$\frac{c}{\Delta} - I(k_{nm}) - \frac{(c/\Delta)^2}{2c/\Delta - I'} \cong 0. \tag{3.10}$$

This is a quadratic equation for Δ, with the solutions

$$\Delta_\pm \equiv \tilde{k}_{nm}^\pm - k_{nm} = c\left\{\frac{1}{I'} + \frac{1}{2I} \pm \left[\left(\frac{1}{I'}\right)^2 + \left(\frac{1}{2I}\right)^2\right]^{\frac{1}{2}}\right\}_{k_{nm}}$$

$$= \left\{\frac{c}{I}\left[\beta + \tfrac{1}{2} \pm (\beta^2 + \tfrac{1}{4})^{\frac{1}{2}}\right]\right\}_{k_{nm}} \tag{3.11a}$$

where

$$\beta \equiv I/I'. \tag{3.11b}$$

The parameter β characterizes the junction widths; it increases with decreasing a_1 or increasing a_2. Thus corresponding to any set of (n, m) there are two distinct resonant wave numbers $\tilde{k}_{nm}^{\pm}$. This is analogous to a coupled spring-mass system with two degrees of freedom (cf. Morse and Ingard [17], p. 63 ff). The separation of the pair of modes is given by

$$\tilde{k}_{nm}^{+} - \tilde{k}_{nm}^{-} \cong 2c\left\{\left[\left(\frac{1}{I'}\right)^2 + \left(\frac{1}{2I}\right)^2\right]^{\frac{1}{2}}\right\}_{k_{nm}}, \tag{3.12}$$

so that as either junction widens the separation increases.

Lastly we point out that in the special case of two equal square basins, $b = b' = l = l'$, there are two equally dominant terms in the series of $F_{\alpha,\alpha}$, $F_{1,1}$, $G_{1,2}$. Hence the coefficient c should be doubled.

Helmholtz modes $(n = m = 0)$

To approximate (3.5), the procedure leading to equation (3.11) can be repeated to get

$$F_{1,1} \cong G_{1,2} \cong \frac{1}{k^2bl}, \quad W \cong \frac{2}{k^2bl} - I'. \tag{3.13}$$

Consequently, (3.5) becomes

$$\frac{1}{k^2bl} - I - \frac{(1/k^2bl)^2}{(1/k^2)(1/2bl) - I'} \cong 0, \qquad k = \tilde{k} \equiv \tilde{k}_{00}. \tag{3.14}$$

Using the fact that I varies with k rather slowly (logarithmically) we may solve k^2 formally from the above quadratic equation

$$(\tilde{k}_{00}^{\pm})^2 \cong \frac{1}{blI}\left[\beta + \tfrac{1}{2} \pm (\beta^2 + \tfrac{1}{4})^{\frac{1}{2}}\right], \tag{3.15}$$

where the right hand side involves the same factor appearing in (3.11a). Equation (3.15) gives two transcendental equations for $\tilde{k}_{00}^{+}$, $\tilde{k}_{00}^{-}$ which may be solved numerically.* Just as in the case of non-Helmholtz modes, the separation of resonant wave numbers $\tilde{k}_{00}^{+}$ and $\tilde{k}_{00}^{-}$ increases with increasing a_1 or a_2 through β.

4. Forced resonant response in coupled basins

General formulas

At a resonant peak, (3.3) and (3.5) imply that the discharges through J and J' are relative maxima with the values:

* The solution to (3.15) gives surprisingly good results compared to more direct computations shown in Section 6.

$$\tilde{Q}_1 \cong 4gA\tilde{\omega}^{-1}, \tag{4.1}$$

$$\tilde{Q}_1' = -\tilde{Q}_2 = \frac{g}{j\tilde{\omega}} M_2 = \frac{4gA}{\tilde{\omega}} \left(\widetilde{\frac{G_{1,2}}{W}} \right). \tag{4.2}$$

Here $\tilde{\omega} = (gh)^{\frac{1}{2}}\tilde{k}$ is a resonant frequency, and the overhead symbol ($\tilde{}$) is used to denote quantities at resonance $k = \tilde{k}$. Let the normalized fluxes be defined as

$$q = \left| \frac{Q_1 \omega}{4gA} \right|, \quad q' = \left| \frac{Q_2 \omega}{4gA} \right|. \tag{4.3a, b}$$

We then have at resonance

$$\tilde{q} \cong 1, \quad \tilde{q}' \cong \left| \widetilde{\frac{G_{1,2}}{W}} \right|. \tag{4.4a, b}$$

It can be shown that (4.4a) holds for a harbor with one basin (B). It provides a simple formula for estimating the mean velocity through the harbor entrance.

The peak (resonant) response of the free surface displacement at any point in the basin B or B' is, from (2.4a, b),

$$\frac{\tilde{\zeta}_B}{2A} \cong -2j \left\{ G_1(x_1, y_1|y_1) - \frac{G_{1,2}}{W} G_2(x_2, y_2|y_2) \right\}_{\tilde{k}}, \tag{4.5a}$$

$$\frac{\tilde{\zeta}_B'}{2A} \cong -2j \left\{ \frac{G_{1,2}}{W} G_1'(x_1', y_1'|y_1') \right\}_{\tilde{k}}. \tag{4.5b}$$

It is convenient to discuss the mean square responses defined by (2.19a) and (2.19b) which give, at resonance:

$$\tilde{\sigma}_B^2 \cong \left[E_1 - 2\frac{G_{1,2}}{W} E_{1,2} + \left(\frac{G_{1,2}}{W} \right)^2 E_2 \right]_{\tilde{k}}, \tag{4.6a}$$

$$\tilde{\sigma}_{B'}^2 \cong \left[\left(\frac{G_{1,2}}{W} \right)^2 E_1' \right]_{\tilde{k}}. \tag{4.6b}$$

Analogous to the simple harmonic oscillator we may define the $\mathcal{Q}$ (quality) factor such that

$$\frac{\hat{k}}{\tilde{k}} = \frac{1}{2\mathcal{Q}}. \tag{4.7}$$

By using eq. (3.6) and straightforward differentiation an explicit formula for $\mathcal{Q}$ may be obtained in the form

$$\begin{aligned} \mathcal{Q} &= \left\{ k^2 bl \left[E_1 - \frac{2G_{1,2}}{W} E_{1,2} + \left(\frac{G_{1,2}}{W} \right)^2 E_2 \right] + k^2 b'l' \left(\frac{G_{1,2}}{W} \right)^2 E_1' \right\}_{\tilde{k}} - \frac{1}{\pi} \\ &= \tilde{k}^2 [bl\tilde{\sigma}_B^2 + b'l'\tilde{\sigma}_{B'}^2] - \frac{1}{\pi} \end{aligned} \tag{4.8}$$

where E_1, $E_{1,2}$, E_2, E_1' are defined in eq. (2.20a, b). We note that the half-width of a resonant peak in the plot of σ_B^2 vs k or $\sigma_{B'}^2$ vs k is $\tilde{k}/\mathcal{Q}$.

If the incident wave is a stationary random process with the one-dimensional wave number spectrum $S(k)$ which varies slowly in k across the width of a peak, then the contributions to the total statistical mean square response $\langle\eta_B^2\rangle$, $\langle\eta_{B'}^2\rangle$ from the peak at $\tilde{k}$ can be estimated by

$$\langle\eta_B^2\rangle_{\tilde{k}} \propto [S\,k\,\sigma_B^2/\mathcal{Q}]_{\tilde{k}}, \qquad \langle\eta_{B'}^2\rangle_{\tilde{k}} \propto [S\,k\,\sigma_{B'}^2/\mathcal{Q}]_{\tilde{k}}. \tag{4.9a, b}$$

These are called the modal mean squares by Miles [7].

The following discussion is again restricted to two equal basins with centered junctions.

Non-Helmholtz modes

Consider first the mean square response in the outer basin B. Keeping the dominant term in each of the series E_1, E_2, $E_{1,2}$ we have

$$E_1 \cong \frac{\varepsilon_n\varepsilon_m}{2}\,\frac{1}{(k_{nm}bl\Delta)^2}, \qquad E_2 \cong E_1, \qquad E_{1,2} \cong E_1 \cos m\pi. \tag{4.10}$$

Upon substituting (4.10) and (3.8) into (4.6a) and using (3.5), it can be shown that

$$\tilde{\sigma}_B^2 \cong \tilde{I}^2 \bigg/ \frac{\varepsilon_n\varepsilon_m}{2}. \tag{4.11}$$

This formula is formally the same as that for a one-basin harbor [7] as if the inner basin B' were absent. Applying (4.11) to $\tilde{k}_{nm}^{\pm}$ separately and subtract, we have by using (2.17)

$$(\tilde{\sigma}_B^2)_- - (\tilde{\sigma}_B^2)_+ \cong \frac{D}{\pi}\ln(\tilde{k}_{nm}^+/\tilde{k}_{nm}^-), \text{ with } D > 0. \tag{4.12}$$

It follows that

$$(\tilde{\sigma}_B^2)_- > (\tilde{\sigma}_B^2)_+. \tag{4.13}$$

Crudely speaking, however, the difference is very small because of the proximity of $\tilde{k}_{nm}^+$ and $\tilde{k}_{nm}^-$ to k_{nm}. For practical purposes, the two peaks may be regarded as nearly equal.

Let us now turn to basin B'. We get by using (3.8) and (3.11a) that

$$\left(\frac{G_{1,2}}{W}\right)_{\pm} = -\frac{1}{\cos m\pi}\,[\beta - \tfrac{1}{2} \pm (\beta^2 + \tfrac{1}{4})^{\frac{1}{2}}]_{k_{nm}}. \tag{4.14}$$

The absolute value of (4.14) gives the normalized discharge through the opening J'; its dependence on the junction widths is plotted in Fig. 4.1. Thus at the higher mode $\tilde{k}_+$, $\tilde{q}'_+$ is strongly affected by junction widths while at $\tilde{k}_-$, $\tilde{q}'_-$ is much less so.

Next we consider $\tilde{\sigma}_{B'}^2$ for the inner basin. Using (4.14) and

$$E_1' \cong c/(\Delta k_{nm}bl)^2, \tag{4.15}$$

and again (3.11a) to eliminate Δ, we obtain

$$(\tilde{\sigma}_{B'}^2)_{\pm} = \left[\left(\frac{G_{1,2}}{W}\right)^2 E_1'\right]_{\pm} \cong \left[\frac{2I^2}{\varepsilon_n\varepsilon_m}\left(\frac{\beta - \frac{1}{2} \pm (\beta^2 + \frac{1}{4})^{\frac{1}{2}}}{\beta + \frac{1}{2} \pm (\beta^2 + \frac{1}{4})^{\frac{1}{2}}}\right)^2\right]_{k_{nm}}. \tag{4.16}$$

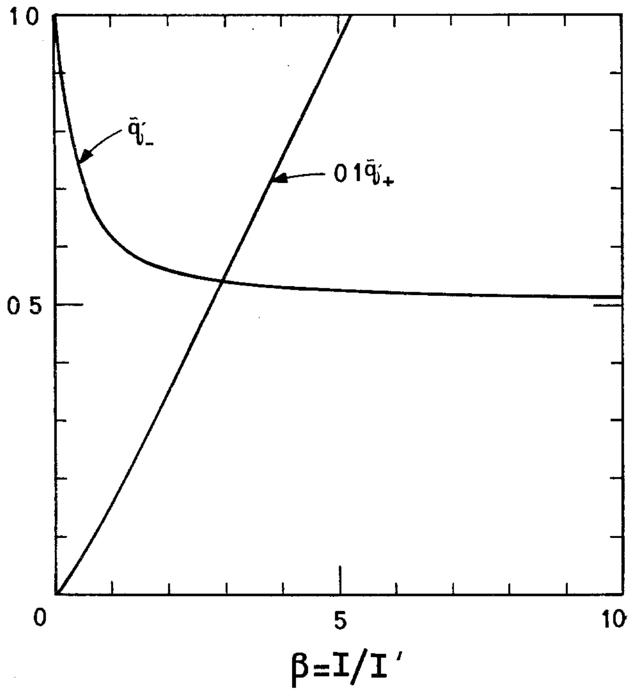

Figure 4.1. Approximate normalized discharge per unit depth through J' at resonance, $q'_{\pm}$ for mode $\tilde{k}_{\pm}$.

After some algebra it may be shown that,

$$(\tilde{\sigma}_{B'}^2)_+ \cong \left\{ \frac{2I^2}{\varepsilon_n \varepsilon_m} \left[\frac{(\beta^2 + \frac{1}{4})^{\frac{1}{2}} - \frac{1}{2}}{\beta} \right]^2 \right\}_{k_{nm}} \tag{4.17}$$

and

$$(\tilde{\sigma}_{B'}^2)_- = \left\{ \frac{2I^2}{\varepsilon_n \varepsilon_m} \left[\frac{\beta}{(\beta^2 + \frac{1}{4})^{\frac{1}{2}} - \frac{1}{2}} \right]^2 \right\}_{k_{nm}}, \tag{4.18}$$

hence

$$\frac{(\tilde{\sigma}_{B'}^2)_+}{(\tilde{\sigma}_B^2)_+} \cong \frac{(\tilde{\sigma}_B^2)_-}{(\tilde{\sigma}_{B'}^2)_-} \cong \left[\frac{(\beta^2 + \frac{1}{4})^{\frac{1}{2}} - \frac{1}{2}}{\beta} \right]^2_{k_{nm}}. \tag{4.19}$$

Now the right hand side of (4.19) is always less than unity, varying from 0 to unity as β increases from 0 to ∞. Hence we obtain the important result on the ordering of the resonant peaks,

$$(\tilde{\sigma}_{B'}^2)_- > (\tilde{\sigma}_B^2)_- > (\tilde{\sigma}_B^2)_+ > (\tilde{\sigma}_{B'}^2)_+. \tag{4.20}$$

This is consistent with the numerical result in [10] and [11], for two equal circular harbors. Thus either by narrowing J' or by widening J, β decreases so that for the lower mode the ratio of inner/outer basin peak responses increases and the inner basin becomes the less protected. This may be regarded as an extension of the harbor paradox of the one-basin case [7]. On the other hand, widening J' or narrowing J has the effect of equalizing the peak responses in both basins, a resonable result in view of the stronger coupling between the basins.

The quality factor can be deduced by using (4.11) and (4.16) in (4.8) with the result

$$\mathcal{Q}_{\pm} \cong \{k^2 bl(\tilde{\sigma}_B^2)_{\pm}[2\beta^2 + \tfrac{1}{2} \mp (\beta^2 + \tfrac{1}{4})^{\frac{1}{2}}]\beta^{-2}\}_{k_{nm}}. \tag{4.21}$$

The modal statistical mean squares are

$$\langle \eta_B^2 \rangle_{\pm} \approx \langle \eta_{B'}^2 \rangle_{\mp} \propto \left[\frac{S}{kbl} f_{\pm}(\beta) \right]_{k_{nm}} \tag{4.22}$$

where

$$f_{\pm}(\beta) \equiv \frac{\beta^2}{2\beta^2 + \frac{1}{2} \mp (\beta^2 + \frac{1}{4})^{\frac{1}{2}}}. \tag{4.23}$$

We remark that the first factor S/kbl in (4.22) is the statistical modal-mean-square for the one-basin harbor, which is independent of the width of the harbor mouth a_1. This is part of the harbor paradox [7]. Here the statistical mean squares of the two basins are affected by the two junction widths through the factor $f_{\pm}(\beta)$, defined in (4.23), which is plotted in Fig. 4.2. Thus by narrowing J' or widening J, β decreases so that $\langle \eta_B^2 \rangle_+$ and $\langle \eta_{B'}^2 \rangle_-$ increases to the value of the one-basin harbor while $\langle \eta_B^2 \rangle_-$ and $\langle \eta_{B'}^2 \rangle_+$ decreases to zero. It is interesting that $\langle \eta_B^2 \rangle_+$ and $\langle \eta_{B'}^2 \rangle_-$ are essentially equal and likewise for $\langle \eta_B^2 \rangle_-$ and

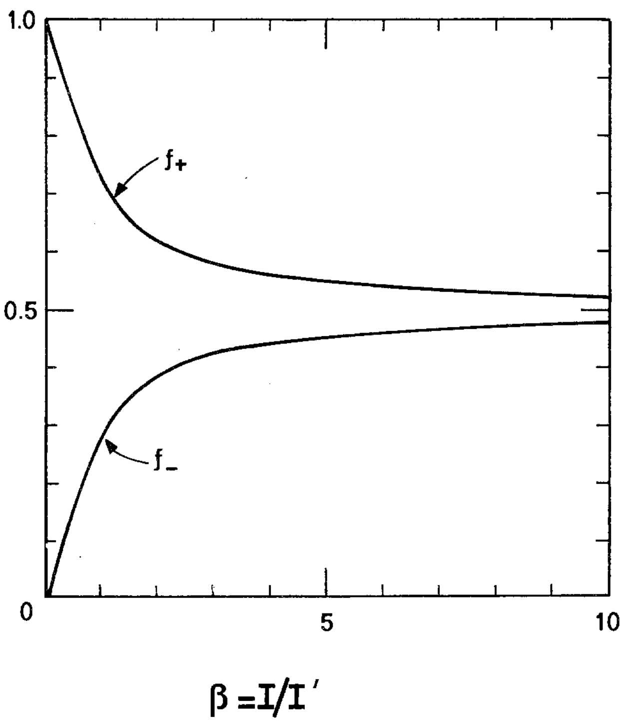

Figure 4.2. Approximate dependence of modal statistical mean square responses of non-Helmholtz modes on junction widths, $\{\langle \eta_B^2 \rangle_+, \langle \eta_{B'}^2 \rangle_-\}(S/kbl)^{-1} = f_+$ and $\{\langle \eta_B^2 \rangle_-, \langle \eta_{B'}^2 \rangle_+\}(S/kbl)^{-1} = f_-$.

$\langle\eta_{B'}^2\rangle_+$. Note that $f_+ + f_- = 1$ and that the two curves in Fig. 4.2 are symmetrical about $f = \frac{1}{2}$.

Helmholtz modes

In an analogous manner the Helmholtz-mode response can be analyzed. We merely point out that (4.11), (4.14) and (4.16) still hold formally, except that k_{nm} must be replaced by $\tilde{k}_{00}^{\pm}$, the solutions to the transcendental equation (3.15). In terms of the local response (cf. (4.5)) the statement corresponding to (4.19) is

$$(\zeta_{B'}/\zeta_B)_\pm = \{\mp [(\beta^2 + \tfrac{1}{4})^{\frac{1}{2}} + \tfrac{1}{2}]\beta^{-1}\}_{\tilde{k}^\pm{}_{00}} \tag{4.24}$$

so that

$$-1 < (\zeta_{B'}/\zeta_B)_+ < 0 \text{ and } 0 < (\zeta_{B'}/\zeta_B)_- < 1. \tag{4.25}$$

It can be seen that the two basins are out of phase for the higher mode $\tilde{k}_{00}^+$ and in phase for the lower mode $\tilde{k}_{00}^-$, though within each basin the free surface rises and falls in unison.

Lastly the modal statistical mean squares are

$$\langle\eta_B^2\rangle_\pm \propto \left[\frac{S\sigma_B^2 k}{\mathscr{Q}}\right]_{\tilde{k}} =$$

$$= \left\{\frac{S\sqrt{I}}{\sqrt{bl}} \; \frac{\beta^2}{[\beta + \frac{1}{2} \pm (\beta^2 + \frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}[2\beta^2 + \frac{1}{2} \mp (\beta^2 + \frac{1}{4})^{\frac{1}{2}}]}\right\}_{\tilde{k}^\pm{}_{00}}, \tag{4.26a}$$

$$\langle\eta_{B'}^2\rangle_\pm \propto \left[\frac{S\sigma_B^2 k}{\mathscr{Q}} \; \frac{\sigma_{B'}^2}{\sigma_B^2}\right]_{\tilde{k}} \cong$$

$$\cong \left\{\frac{S\sqrt{I}}{\sqrt{bl}} \; \frac{\beta^2 + \frac{1}{2} \mp (\beta^2 + \frac{1}{4})^{\frac{1}{2}}}{[\beta + \frac{1}{2} \pm (\beta^2 + \frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}[2\beta^2 + \frac{1}{2} \mp (\beta^2 + \frac{1}{4})^{\frac{1}{2}}]}\right\}_{\tilde{k}^\pm{}_{00}}. \tag{4.26b}$$

We point out that the factor $S(I/bl)^{\frac{1}{2}}$ is the modal response of the Helmholtz mode of the one-basin harbor [7], which increases as J narrow. The modification factors due to the coupling between basins are plotted in Fig. 4.3. Thus for the higher mode where the two basins are opposite in phase, the modal response of the outer basin decreases (curve 1) with increasing β while that of the inner basin increases to a relatively small value then decreases (curve 2). For the lower mode where the two basins are in phase, the outer basin response increases (curve 3) while that of the inner basin (curve 4) decreases to the same asymptotic value $S(I/2bl)^{\frac{1}{2}}$. In this limit the two basins oscillate as a single basin with twice the length because of the comparatively strong coupling between them.

5. The effect of partitions of finite thickness

In the preceeding analysis we assumed that the partitions between the ocean and the outer basin, and between the two basins were thin. For a harbor with a single basin, Carrier, Shaw and Miyata [8] found in numerical examples that the effect of finite length of the entry channel, i.e., finite thickness of the barriers near junction J, is qualitatively the same as if the width between the thin-walled barriers is reduced. This result can be generalized

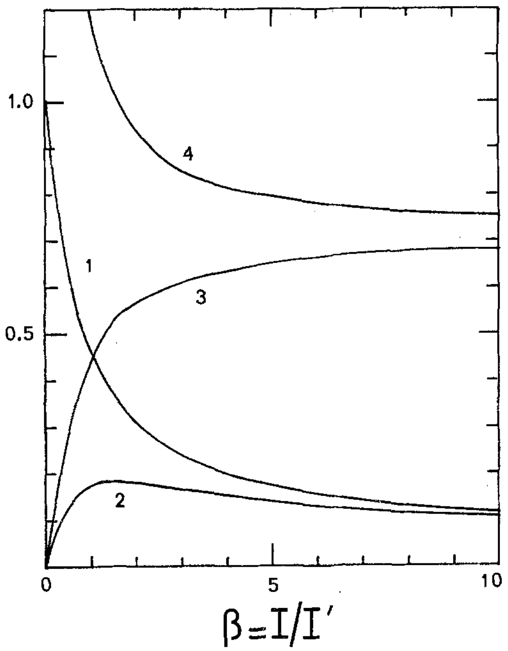

Figure 4.3. Approximate dependence of modal statistical mean square responses of Helmholtz modes on junction widths. Ordinates of curves 1, 2, 3, 4 are respectively
$[\langle\eta_B^2\rangle_+, \langle\eta_{B'}^2\rangle_+]/(S[I/bl]^{\frac{1}{2}})_+, [\langle\eta_B^2\rangle_-, \langle\eta_{B'}^2\rangle_-]/(S[I/bl]^{\frac{1}{2}})_-$.

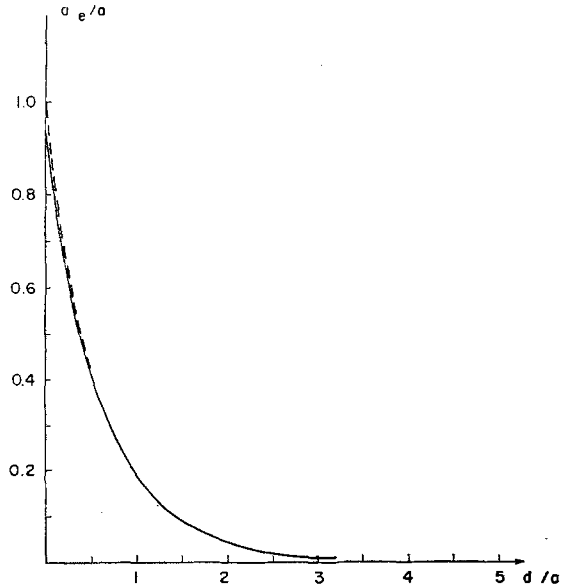

Figure 5.1. The ratio of effective-width-to-actual-width of a junction as a function of the thickness-to-width ratio ——. Eq. (5.1) - - - -.

analytically by matched asymptotics for several junctions. Indeed, all the results obtained for thin-walled junctions can be reinterpreted for thick walled junctions if an effective width a_e is introduced to replace the actual width a. More specifically, if the thickness of the junction ($2d$) is of the same order of magnitude as the width ($2a$) then the ratio a_e/a can be expressed in terms of elliptic integrals, see [14] and [18]. We omit the details and only plot the ratio a_e/a in Figure 5.1 as a function of the thickness-to-width ratio d/a. Clearly, increasing the junction thickness amounts to reduction of junction width. A practical approximation for $d/a > 0.5$ is [14]

$$\frac{a_e}{a} \simeq \frac{8}{\pi} \exp\left[-\left(\frac{\pi d}{2a} + 1\right)\right] \tag{5.1}$$

which is surprisingly close to the more exact result, see the dashed line in Fig. 5.1.

6. Numerical results

In order to confirm the analytical conclusions, we have performed computations for the root-mean-square responses σ_B and $\sigma_{B'}$ directly from (2.19a, b) without further approximation. In examining the effect of finite junction thickness the results discussed in Section 5 are utilized.

The two basins are assumed to be square and equal and the junctions centered. The width of the entrance J is chosen to be $2a_1/b = 3.10^{-2}$. In Fig. 6.1, the interbasin junction J' is taken to be of the same width as J, both having zero thickness. Within the computed range $0 \leqq kb \leqq 8$, the distinct natural modes of one basin B or B' are

$$k_{01}b = \pi = 3.1415, \quad k_{02}b = k_{20}b = 2\pi = 5.2833, \quad k_{21}b = \sqrt{5}\pi = 7.0248,$$

which correspond to the second, third, and fourth pairs of peaks, respectively, the first pair being the Helmholtz modes. Clearly the ordering of the first three pairs of peaks obey (4.20). The ordering of the last pair of peaks is only in partial agreement with (4.20), because for the higher modes the parameter ka which is supposed to be small in the approximation is beginning to become appreciable (> 0.107). As we shall see, however, (4.20) applies even for the fourth pair if ka is reduced.

In Fig. 6.2 the width of J' is increased so that $a_2 = 4a_1$; the thicknesses of both J and J' are still zero. Let us examine the lowest three pairs of peaks. In comparison with Fig. 6.1, the separation between a pair $\tilde{k}_\pm$ is indeed increased, and for the same mode the difference between the responses of the basins is reduced, as is predicted analytically. Note that for the fourth and highest pair of peaks the ordering rule (4.20) has deteriorated further for now ka_2 is greater than 0.426.

In Fig. 6.3 we keep $a_1 = a_2$ but increase the thickness d_2 of J' from zero to $d_2 = 2a_2$. In accordance with the results in Section 5, this is equivalent to a reduction in the actual width a_2 of J'. In comparison with Fig. 6.1 we see that now even the highest pair of peaks obey the ordering rule (4.20), since ka_2 is reduced to ~ 0.027. Moreover, as predicted the separation between pairs of peaks at $\tilde{k}_\pm$ decreases while for the same mode the difference between the resonant responses of the basins is increased.

In Figs. 6.1 to 6.3 the points where $\sigma_B^2 = \sigma_{B'}^2$ are marked by vertical dashes the corresponding wave numbers are the real zeroes of W (cf. (2.16)). This is proven as follows.

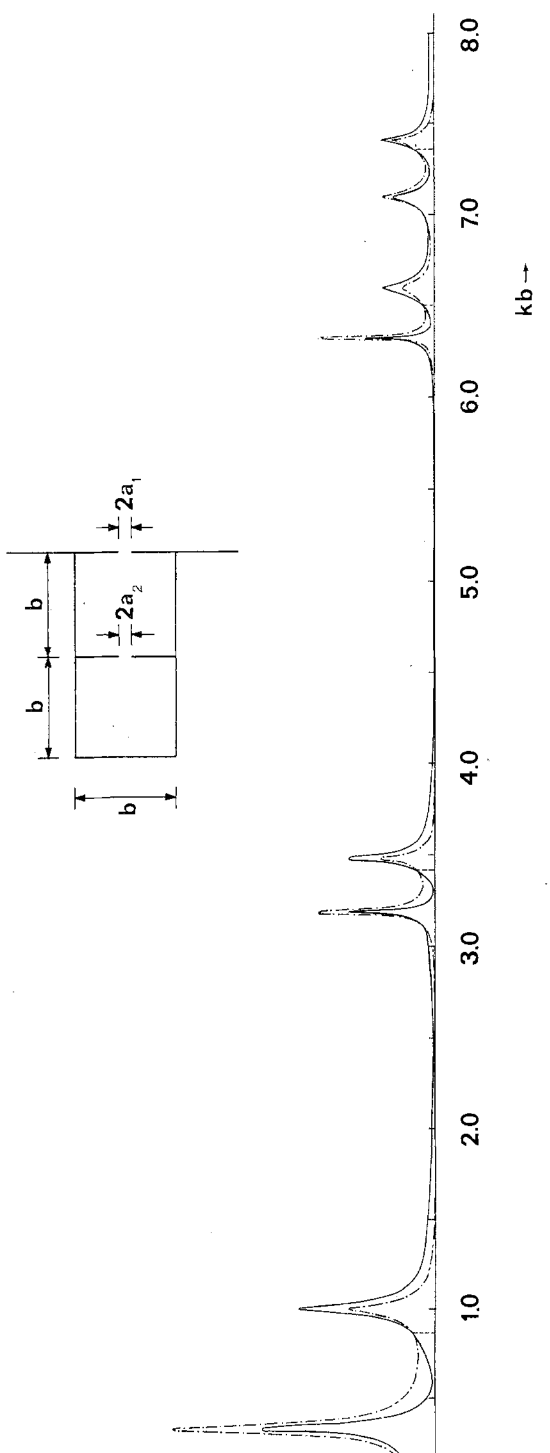

Figure 6.1. Root-mean-square responses of two square and identical basins, as functions of kb. $2a_1/b = 3 \times 10^{-2}$, $a_1 = a_2$. Outer basin (——), Inner basin (-·-·-·-).

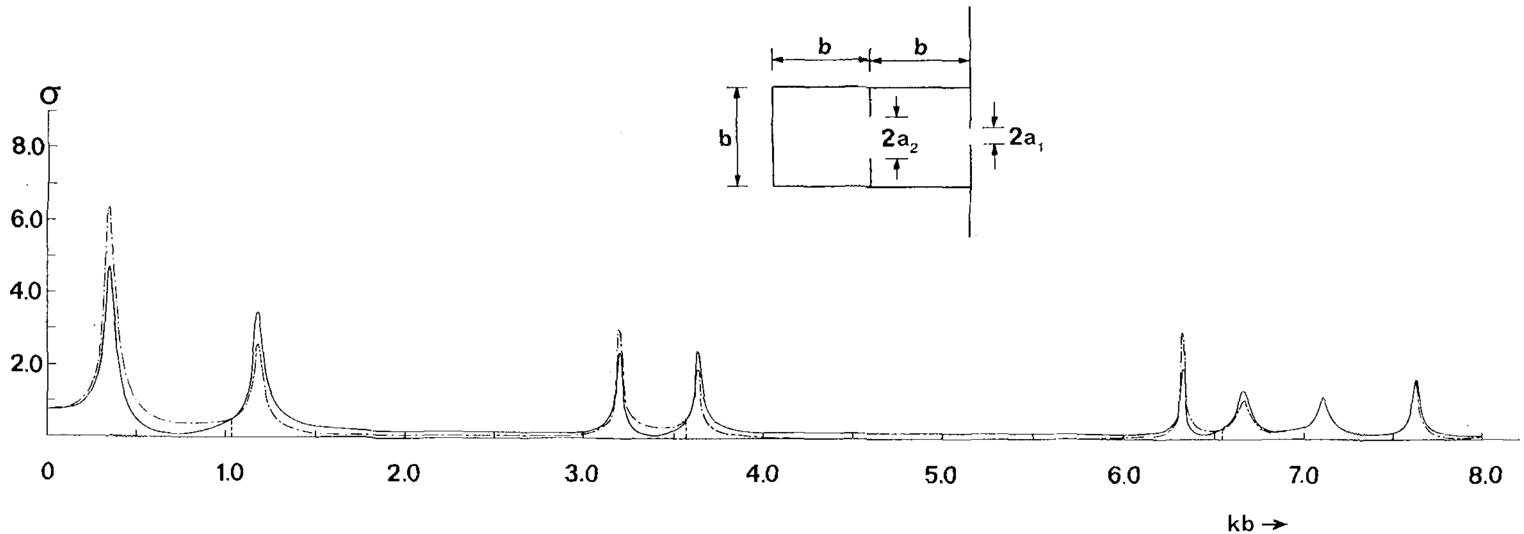

Figure 6.2. Root-mean-square responses of two square and identical basins, as functions of kb. $2a_1/b = 3 \times 10^{-2}$, $a_2 = 4a_1$. Outer basin (———), Inner basin (-·-·-·-).

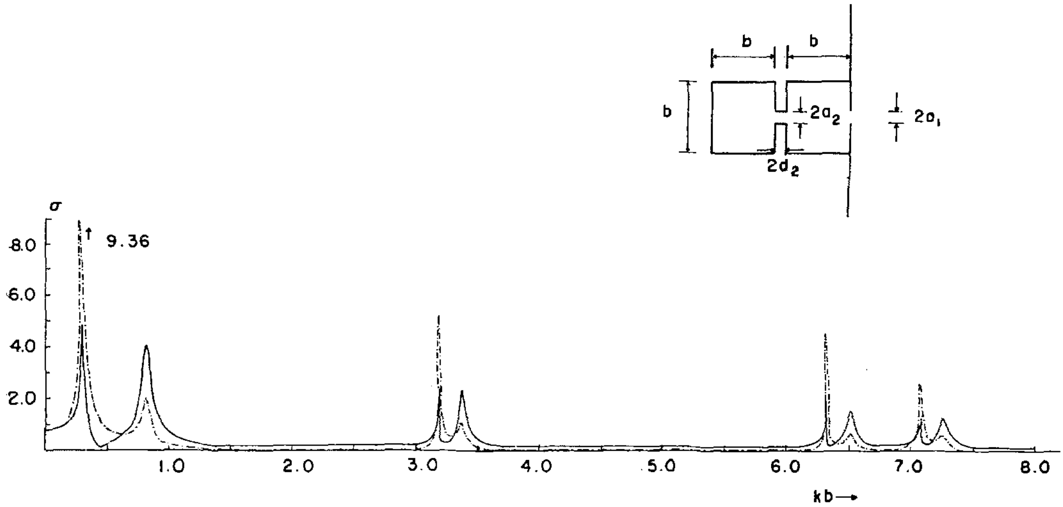

Figure 6.3. Root-mean-square responses of two square and identical basins, as functions of kb. $2a_1/b = 3 \times 10^{-2}$, $a_1 = a_2$, $d_2 = 2a_2$, $d_1 = 0$. Outer basin (———), Inner basin (-·-·-·-).

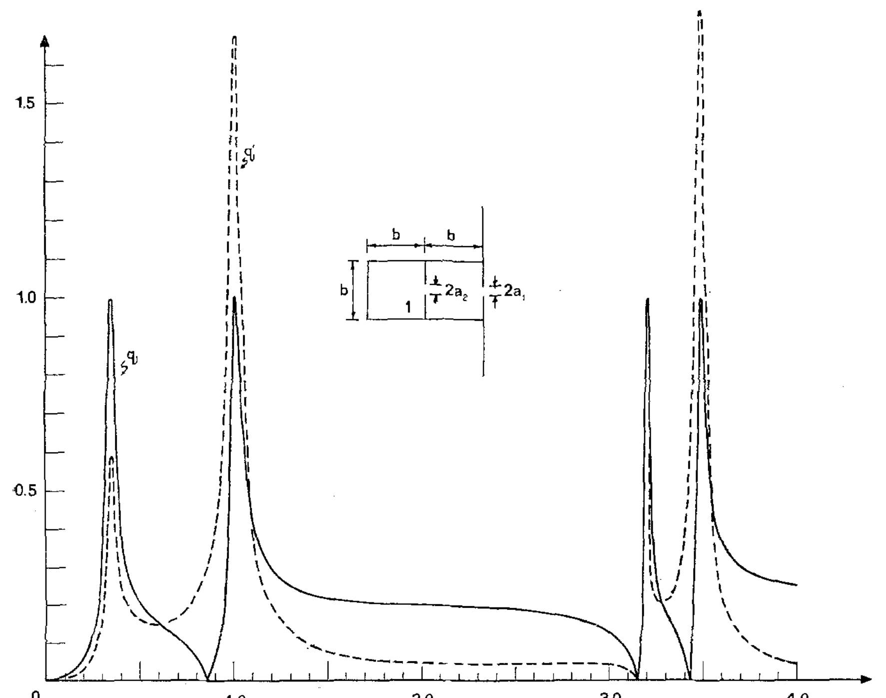

Figure 6.4. Normalized discharges per unit depth through junctions J and J'. $|q|$: ———, $|q'|$: – – – –. $a_1 = a_2$, $2a_1/b = 3 \times 10^{-2}$.

As W approaches zero, (2.15a, b) imply that

$$\frac{\pi M_1}{2A} \sim \frac{W}{G_{1,2}^2} \to 0, \quad M_2 \neq 0,$$

which means that the mouth J is closed but flow can be allowed through J'. In other words, the real zeroes of W are the eigen wave numbers of the two coupled basins unconnected to the sea. From (2.19) it is easy to show that at these values of k, $\sigma_B^2 \sim \frac{1}{4}E_2/G_{1,2}^2 \sim \sigma_{B'}^2$. In Fig. 6.4 the normalized fluxes through the two junctions are plotted for $a_1 = a_2$. The peak values of q through J are indeed 1. Furthermore it can be noted that the values of kb where $q = 0$ exactly correspond to the zeroes of W. Thus at these frequencies pressure is transmitted through the harbor entrance but mass is not.

7. Concluding remarks

By employing the method of matched asymptotics, and performing the partial summation of a series, we have obtained an analytic solution for a harbor with two rectangular basins coupled by narrow junctions. For the special case of two equal basins with their line of centers normal to the coast the resonant spectrum and response are approximately analyzed in detail. Several interesting results suggested by numerical experiments of [10] and [11] are confirmed. While these results should be modified for other geometries (unequal basins or other junction positions or basins connected to narrow channels), similar method of solution and analysis can be conveniently applied [14]. Needless to say for arbitrary basin or coastline geometry, recourse to numerical method is necessary. Nevertheless analytical effort can provide insight and guidance to these more complex problems.

An interesting aspect which we did not study here is the possibility of oscillatory energy transfer between the basins in a transient problem. This is anticipated by analogy to the simple oscillator with two degrees of freedom [17], p. 67.

Finally, it should be noted that the present theory ignores certain facts of reality, notably, boundary friction losses at the bottom and at the junction due to flow separation. These effects have an important influence on resonance by augmenting the total damping and thereby restricting the validity of the paradox. Some studies of the entrance loss have been made by Ünlüata and Mei [19] and Miles and Lee [9].

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Appendix A. Inner expansion of the source solution

The typical source solution is given by (2.4). Omitting the subscript α, it can be rewritten as

$$G = -\sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-n\pi x/b} Y_n(y) Y_n(y) + \frac{\cos k(x-l)}{kb} + \sum_{n=1}^{\infty} \left[X_n(x) + \frac{2}{n\pi} e^{-n\pi x/b} \right] Y_n(y) Y_n(y).$$

The first series, obtained from the original by approximating X_n for large n, will be abbreviated by $\tilde{F}$. The residual series will be denoted by F; it has terms which die out as (n^{-3}) and can be efficiently computed. The series $\tilde{F}$ can be summed exactly to give

$$\tilde{F} = \frac{1}{\pi} \ln (|1 - e^{-Z}|^2 |1 - e^{-Z^*}|^2)$$

where

$$Z = \frac{\pi}{b}(x + i(y - y)), \quad Z^* = \frac{\pi}{b}(x + i(y + y)).$$

It may be pointed out that Z is the normalized complex distance from the source to the field point and Z^* is the complex distance from the image source (with respect to the side $y = 0$) to the field point.

We now derive the inner expansion of G for small r/b. Let us assume that the source point is far from a corner: $y \neq 0$ or b/π. By Taylor expansion we get

$$\tilde{F} = \frac{1}{\pi} \ln \left(\frac{2\pi r}{b} \sin \frac{\pi y}{b}\right) \cdot \left(1 + O\left(\frac{r}{b}\right)\right)$$

Note that as $r \to 0$, $\tilde{F} \to (1/\pi) \ln r$, hence the corresponding mass flux through an infinitesimal half circle centered at the source point of the side of $x > 0$ is unity. This shows that $\tilde{F}$ accounts for all the singular behavior of G, hence F is regular. The leading term in the inner approximation is then

$$G \cong \frac{1}{\pi} \ln \left(\frac{2\pi r}{b} \sin \frac{\pi y}{b}\right) + F(0, y|y). \tag{A.1}$$

If the harbor entrance is very close to one corner then one should account for the effect of the image source. The modification is straightforward but will not be considered here.

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